Stability Analysis of Time Domain Discontinuous Galerkin H-Φ Method for Eddy Current Simulations

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A rigorous stability analysis of the previously published semi-explicit time domain Discontinuous Galerkin (DG) H-Φ approach for eddy current simulations is presented. The considered DG finite element method (FEM) enables explicit time stepping in electrically conducting regions and eliminates the need for solving large sparse ill-conditioned equation systems. The considered method utilizes the magnetic scalar potential in electrically non-conducting regions computed by using the nodal finite elements. The theoretical stability limit of the considered semi-explicit time domain approach is obtained by using the Z-transform of the discrete time domain DG-FEM equations and by performing an eigenvalue analysis of the underlying elemental DG-FEM matrices. The obtained results are tested on a simple 3-D example and an excellent agreement between the theoretical stability limit and the empirically obtained values was found.

***Index Terms*—Time domain method, eddy current, discontinuous Galerkin, finite element method, and numerical stability.**

I. Introduction

ACCURATE EDDY CURRENT simulations are of paramount importance for modern design of power devices such as power and distribution transformers, circuit breakers, switchgears, and reactors. Already in the early seventies, this was recognized by a wide scientific community resulting in numerous publications considering eddy current field formulations [1], [2] and the corresponding numerical solution methods [3], [4].

The most widely used numerical methods for solving eddy current problems are presently vector FEM frequency domain (FD) approaches based on the T-Ω field formulation, such as for example the solvers presented in [5], [6].

To overcome considerable numerical problems of the modern FD eddy current solvers (such as notoriously ill-conditioned curl-curl matrix, inaccurate treatment of nonlinear materials, and expensive matrix preconditioning for iterative solvers in terms of CPU-time and memory, [3], [4]) one needs to perform the simulations in time domain with an explicit time stepping, i.e. with a time stepping that does not require a solution of a large linear equation system in each time step. This is possible to achieve by using the DG-FEM that numerically decouples finite elements enabling the use of non-conforming meshes [7] and explicit time stepping [8].

In the publication [9] a 2-D eddy current solver based on the A-A field formulation and time domain DG-FEM is described. As opposed to this work that deals with an implicit scheme and ill-conditioned curl-curl matrix in nonconductive domain, a novel 3-D eddy current time domain DG-FEM solver based on the H-Φ field formulation was recently presented [10]. The semi-explicit H-Φ eddy current solver utilizes the nodal FEM in the implicit nonconductive part for computing the magnetic scalar potential thus removing completely the ill-conditioned curl-curl matrix [10].

The recent publication [10] describes the 3-D time domain DG-FEM for eddy current simulations in details without considering its stability analysis. The corresponding stability analysis is the main scientific contribution of this paper.

II. Stability Analysis

In the recent publication [10] the theoretical and implementation details of the novel time domain DG-FEM H-Φ eddy current solver were presented. For this reason only the equation relevant for stability analysis are repeated here.

According to [10], the explicit time stepping of the considered DG-FEM H-Φ is performed according to the following equations based on the forward difference approach

$$\vec{E}^{(k)} \to \vec{H}^{(k+1)} \to \vec{E}^{(k+1)} \to \vec{H}^{(k+2)} \tag{1}$$

$$\begin{aligned}\left\{H_e^{(k+1)}\right\} = \left\{H_e^{(k)}\right\} - \Delta t\left[A_e^2\right]^{-1}\Bigg(\left[A_e^1\right]\left\{E_e^{(k)}\right\} + \\ +\frac{1}{2}\sum_{f=1}^{4}\left[\left[B_{ep}^f\right]\left\{E_e^{+(k)}\right\} - \left[B_{ee}^f\right]\left\{E_e^{(k)}\right\}\right]\Bigg)\end{aligned} \tag{2}$$

$$\begin{aligned}\left\{E_e^{(k+1)}\right\} = \left[A_e^3\right]^{-1}\Bigg(\left[A_e^1\right]\left\{H_e^{(k+1)}\right\} + \\ +\frac{1}{2}\sum_{f=1}^{4}\left[\left[B_{ep}^f\right]\left\{H_e^{+(k+1)}\right\} - \left[B_{ee}^f\right]\left\{H_e^{(k+1)}\right\}\right]\Bigg)\end{aligned} \tag{3}$$

where the matrix entries in (2) and (3) can be computed as follows

$$A_e^1(i,j) = \iiint_{(\Omega_e)} \vec{N}_i \cdot \left(\nabla \times \vec{N}_j\right) dV \text{, } A_e^2(i,j) = \iiint_{(\Omega_e)} \mu \vec{N}_i \cdot \vec{N}_j dV \tag{4}$$

$$A_e^3(i,j) = \iiint_{(\Omega_e)} \sigma \vec{N}_i \cdot \vec{N}_j dV \text{, } B_{ep}^f(f,i,j) = \iint_{(\Delta_e^f)} \vec{N}_i \cdot \left(\vec{n} \times \vec{N}_j^+\right) dS \tag{5}$$

$$B_{ee}^f(f,i,j) = \iint_{(\Delta_e^f)} \vec{N}_i \cdot \left(\vec{n} \times \vec{N}_j\right) dS \tag{6}$$

where Δ_e^f represents the fth face of the eth tetrahedron Ω_e, "+" denotes the field value on the other side of the interface, i.e. outside of the considered tetrahedron, and $\vec{N}_i$ is the vector shape function related to the ith edge of the tetrahedron [4]. The integrals (4) – (6) in this study were numerically evaluated by using the well-known Gauss quadrature.

Equation (2) and (3) can be combined and written in the following compact form

$$\left\{H_e^{(k+1)}\right\} - \left\{H_e^{(k)}\right\} = -\Delta t\left[M_e\right]\left\{H_e^{(k)}\right\} + \left\{b_e^{(k)}\right\} \tag{7}$$

where the source vector $b_e^{(k)}$ of the time step "k" encompasses the field values outside of the considered tetrahedron.

By applying the Z-transform to the discrete system (7) and by taking into account that only the homogenous part of the obtained equation is relevant for the system stability, the following homogenous equation is obtained [11]

$$(z-1+\Delta t\lambda)\{\underline{H}_e\}=0 \tag{8}$$

where z is the complex variable, Δt is the time step, and λ is an eigenvalue of the matrix $[M_e]$.

According to the theory of discrete systems [11], the system is stable if the complex roots of the characteristic equation (8) are located in the complex plane within the unit circle. This yields the following stability limit

$$\Delta t_{\max} \le \frac{2}{\rho([M_e])} \tag{9}$$

where $\rho([M_e])$ is the spectral radius of the matrix $[M_e]$, i.e. $\rho([M_e])=\max(|\lambda_1|,|\lambda_2|,\ldots,|\lambda_n|)$ is the maximal absolute eigenvalue of the matrix $[M_e]$. The details of derivation of Equation (9) will be given in the full paper.

III. Results and Conclusions

The obtained stability condition (9) can be used to evaluate the influence of the mesh quality on the integration time step. The aspect ratio of a tetrahedron was gradually increased, as shown in Fig. 1 (top), and the corresponding stability limit was computed. The obtained results, shown in Fig. 1 (bottom), reveal the following important power function $\Delta t_{max}=A\cdot l_{edge}^{\delta}$, where the constant A depends on the element's aspect ratio and material properties and δ depends on the interpolation order of the tetrahedron.

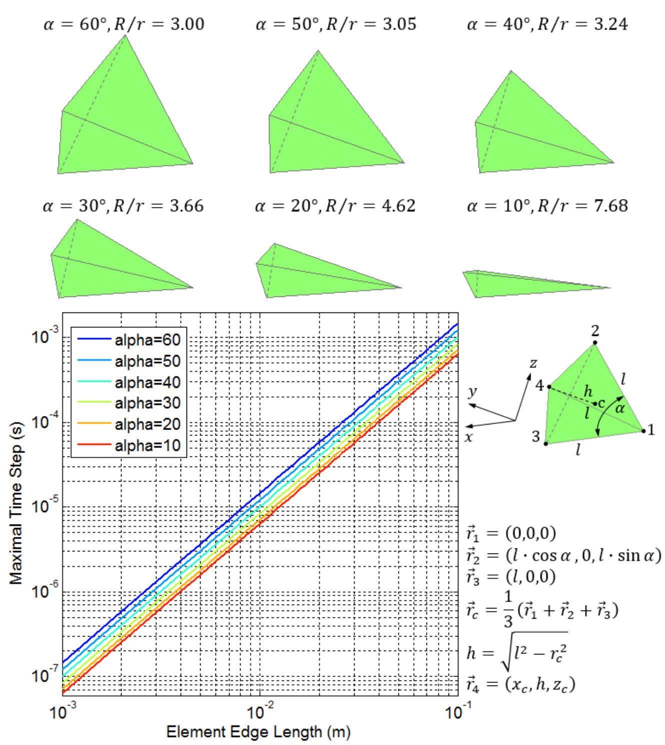

Fig. 1. Stability condition for tetrahedrons of different quality measured by the aspect ratio (the radius ratio between the circumscribed and inscribed sphere of the tetrahedron) is presented. The obtained results reveal the following dependence of the stability limit from the edge length $\Delta t_{max}=A\cdot l_{edge}^{\delta}$. The numerical results were obtained for the following material data $\sigma=3.5\cdot10^6 S/m$, $\mu_r=1$.

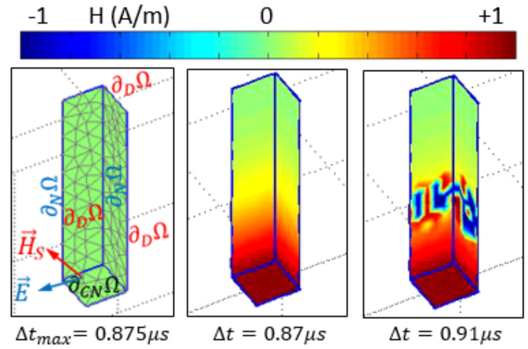

Fig. 2. A simple 3-D eddy current example is presented (left). A rectangular block of a conductive material with the material properties $\sigma=3.5\cdot10^6 S/m$, $\mu_r=1$ is considered. The source magnetic field is applied on the front surface ($\partial_{CN}\Omega$) and has the amplitude 1A/m and frequency 50Hz. The obtained magnetic field at the moment of time 5ms are depicted with the time step slightly below the stability limit (middle) and slightly above it (right). The effect of the lost stability is evident.

To practically test the validity of the theoretically obtained stability condition (9) a simple 3-D eddy current problem presented in Fig. 2 was defined. It is evident in Fig. 2 that the stability condition (9) is practically confirmed.

In the full paper will be given all the theoretical details related to Equations (7)-(9), as well as the numerical results of realistic 3-D eddy current problems with complex geometry.

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